Generalized squeezing operators, bipartite Wigner functions, and entanglement via Wehrl's entropy functionals

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We introduce a new class of unitary transformations based on the $\mathfrak{su}(1,1)$ Lie algebra that generalizes, for certain particular representations of its generators, well-known squeezing transformations in quantum optics. To illustrate our results, we focus on the two-mode bosonic representation and show how the parametric amplifier model can be modified in order to generate such a generalized squeezing operator. Furthermore, we obtain a general expression for the bipartite Wigner function which allows us to identify two distinct sources of entanglement, here labelled by dynamical and kinematical entanglement. We also establish a quantitative estimate of entanglement for bipartite systems through some basic definitions of entropy functionals in continuous phase-space representations.

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I. INTRODUCTION

In the last few years, physics has experienced the appearance of two relatively young branches with strong appeal in both theoretical and experimental aspects. Labelled by Quantum Information and Quantum Computation, these branches apparently entangled have attracted since then a lot of attention from researchers working in well-established areas in physics (such as, for instance, solid state physics, nuclear physics, high energy physics, general relativity, and cosmology [1]). The interdisciplinarity provided by Quantum Information and Quantum Computation is basically focussed upon fundamental physical concepts that constitute the cornerstones of quantum mechanics. Hence, concepts related to nonclassical states, superposition principle, entanglement, phase-space representations, quantum teleportation, quantum key distribution, among others, represent nowadays common words in many scientific papers covering different areas of knowledge in physics.

In particular, let us restrict our attention to quantum information theory and its description by quantum continuous variables [2], where entanglement effects can be efficiently produced in laboratory through the adequate manipulation of continuous quadrature amplitudes of the quantized electromagnetic field. In this promising scenario, the squeezed light [3, 4] has a prominent role in the experimental implementation of continuous-variable entanglement, since the degree of imperfection in entanglement-based quantum protocols depends on 'the amount of squeezing of the laser light involved'. From a conceptual point of view, some questions related to the entanglement measures (or separability criteria) still remain open in the specialized literature [5]. For instance, entanglement of formation [6] and concurrence [7] are now widely accepted as entanglement measures for the two-qubit case. However, it is worth noticing that different approaches to this problem exist which are outlined by means of information-theoretic arguments [8].

The main goal of this paper is to present some contributions to certain specific topics in quantum information theory that allow us to go further in our comprehension on the entanglement process in ideal bipartite systems (the interaction with any dissipative environment [9] is discarded in a first moment). For this purpose, we first construct a general family of unitary transformations associated with the $\mathfrak{su}(1,1)$ Lie algebra generators that generalizes – if one considers the one- and two-mode bosonic representations of its generators – two well-known expressions of squeezing operators [3]. Following, we study a modified version of the parametric amplifier [10] with emphasis in obtaining the solutions of the Heisenberg equations and its respective time-evolution operator. In particular, we show the efficacy of this model in generating the generalized two-mode squeezing operator through its connection with the time-evolution operator.

The next step then consists in performing a preliminary study via Wigner function on the qualitative aspects of entanglement for a general class of bipartite systems whose dynamics is governed by arbitrary quadratic Hamiltonians. As a by-product of this investigation, we obtain a general integral representation for the bipartite Wigner function that leads us to identify two distinct sources of entanglement (here labelled by dynamical and kinematical entanglement). The last contribution refers to a direct application of the results obtained by Piątek and Leoński [11] for the intermode correlations in continuous phase-space representations, which are based on the Wehrl's approach [12] regarding some definitions of entropy functionals and their inherent properties. In fact,

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we introduce a correlation functional that permits us to measure the degree of entanglement between both parts of the joint system for any initial conditions associated with the Hamiltonian operator. It is important to emphasize that the sequence of topics covered in this work presents an inherent logical consistency that improves our comprehension on the subtle mechanisms associated with the entanglement effects in bipartite systems.

This paper is structured as follows. In Section II, we present a mathematical statement that leads us to construct a general family of unitary transformations associated with the $\mathfrak{su}(1,1)$ Lie algebra where, in particular, the two-mode bosonic representation is emphasized. In Section III, we obtain the solutions of the Heisenberg equations for the parametric amplifier model, and show the subtle link between time-evolution operator and generalized two-mode squeezing operator. The results obtained are then applied in Section IV, within the context of Wigner functions, in order to establish an initial discussion on entanglement for certain groups of bipartite states of the electromagnetic field. Section V is devoted to establish a reasonable measure of entanglement which is based on some specific information-theoretic arguments. Besides, we illustrate our results through two different examples of initially uncoupled bipartite states for the model under investigation. Finally, Section VI contains our summary and conclusions.

II. SQUEEZING OPERATORS ASSOCIATED WITH THE $\mathfrak{su}(1,1)$ LIE ALGEBRA

Let us initially introduce the generators of the $\mathfrak{su}(1,1)$ Lie algebra $\mathbf{K}_{\pm}$ and $\mathbf{K}_0$, which satisfy the following commutation relations:

$$[\mathbf{K}_-, \mathbf{K}_+] = 2\mathbf{K}_0 \quad \text{and} \quad [\mathbf{K}_0, \mathbf{K}_{\pm}] = \pm\mathbf{K}_{\pm} \, .$$

The Casimir operator is defined within this context through the mathematical identity

$$\mathbf{K}^2 := \mathbf{K}_0^2 - (1/2)\{\mathbf{K}_-, \mathbf{K}_+\}$$

with $[\mathbf{K}^2, \mathbf{K}_{\pm}] = [\mathbf{K}^2, \mathbf{K}_0] = 0$, where $\{\mathbf{K}_-, \mathbf{K}_+\}$ represents the anticommutation relation between the operators $\mathbf{K}_-$ and $\mathbf{K}_+$. Furthermore, let us also consider the abstract operator

$$\mathbf{T}(\Omega_{\pm}, \Omega_0) := e^{\Omega_+ \mathbf{K}_+ + \Omega_0 \mathbf{K}_0 + \Omega_- \mathbf{K}_-} \tag{1}$$

written in terms of the arbitrary c-number parameters $\Omega_{\pm}$ and Ω_0, whose generalized normal- and antinormal-order decomposition formulas have already been established in literature [13, 14]. These preliminary considerations on the $\mathfrak{su}(1,1)$ Lie algebra lead us to demonstrate an important composition formula involving the product of two abstract operators, each one being defined as in equation (1) and characterized by a particular set of arbitrary c-number parameters. In fact, this section will deal with some applications of the composition formula with emphasis on unitary transformations which resemble, for certain representations of the generators $\mathbf{K}_{\pm}$ and $\mathbf{K}_0$, the squeezing transformations in quantum optics.

Lemma II.1 *Let $\mathbf{T}(\Omega_{\pm}, \Omega_0)$ and $\mathbf{T}(\Lambda_{\pm}, \Lambda_0)$ be two abstract operators whose functional forms obey equation (1). For a given set of arbitrary c-number parameters $\{\Omega_{\pm}, \Omega_0, \Lambda_{\pm}, \Lambda_0\}$ it is always possible to verify the general composition formula*

$$\mathbf{T}(\Omega_{\pm}, \Omega_0)\mathbf{T}(\Lambda_{\pm}, \Lambda_0) = \mathbf{T}(\Sigma_{\pm}, \Sigma_0), \tag{2}$$

where $\Sigma_{\pm}$ and Σ_0 are solutions of the coupled set of nonlinear equations

$$\frac{A_+ + B_0 B_+ \left[A_0 - A_+ (A_- + B_-)\right]}{B_0 (A_- + B_-)} = \frac{\Sigma_+}{\Sigma_-}, \tag{3a}$$

$$\sqrt{B_0/A_0}\,(A_- + B_-) = (\Sigma_-/\beta)\sinh(\beta), \tag{3b}$$

with $\beta = [(\Sigma_0/2)^2 - \Sigma_+\Sigma_-]^{1/2}$. The c-number functions $A_{\pm}$, A_0, $B_{\pm}$, and B_0, present in the lhs of equations (3a) and (3b), are connected with $\{\Omega_{\pm}, \Omega_0, \Lambda_{\pm}, \Lambda_0\}$ through the identities [14]:

$$A_{\pm} = \frac{(\Omega_{\pm}/\phi)\sinh(\phi)}{\cosh(\phi) - (\Omega_0/2\phi)\sinh(\phi)}, \tag{4a}$$

$$A_0 = \left[\cosh(\phi) - (\Omega_0/2\phi)\sinh(\phi)\right]^{-2}, \tag{4b}$$

$$B_{\pm} = \frac{(\Lambda_{\pm}/\theta)\sinh(\theta)}{\cosh(\theta) + (\Lambda_0/2\theta)\sinh(\theta)}, \tag{4c}$$

$$B_0 = \left[\cosh(\theta) + (\Lambda_0/2\theta)\sinh(\theta)\right]^2, \tag{4d}$$

for

$$\phi = [(\Omega_0/2)^2 - \Omega_+\Omega_-]^{1/2},$$
$$\theta = [(\Lambda_0/2)^2 - \Lambda_+\Lambda_-]^{1/2}.$$

Proof II.1 *Firstly, we apply the generalized normal- and antinormal-order decomposition formulas established by Ban [14] for exponential functions of the generators of $\mathfrak{su}(1,1)$ Lie algebra to the abstract operators $\mathbf{T}(\Omega_{\pm}, \Omega_0)$ and $\mathbf{T}(\Lambda_{\pm}, \Lambda_0)$, that is,*

$$\mathbf{T}(\Omega_{\pm}, \Omega_0) = e^{A_+\mathbf{K}_+} e^{\ln(A_0)\mathbf{K}_0} e^{A_-\mathbf{K}_-} \tag{5}$$

and

$$\mathbf{T}(\Lambda_{\pm}, \Lambda_0) = e^{B_-\mathbf{K}_-} e^{\ln(B_0)\mathbf{K}_0} e^{B_+\mathbf{K}_+}, \tag{6}$$

where the c-number functions $A_{\pm}$, $B_{\pm}$, A_0, and B_0 were defined by means of the identities (4a)-(4d). In this way, the product $\mathbf{T}(\Omega_{\pm}, \Omega_0)\mathbf{T}(\Lambda_{\pm}, \Lambda_0)$ can be expressed as

$$\begin{aligned}\mathbf{T}(\Omega_{\pm}, \Omega_0)\mathbf{T}(\Lambda_{\pm}, \Lambda_0) &= e^{A_+\mathbf{K}_+} e^{\ln(A_0)\mathbf{K}_0} e^{(A_- + B_-)\mathbf{K}_-} \\ &\quad \times e^{\ln(B_0)\mathbf{K}_0} e^{B_+\mathbf{K}_+} \\ &= e^{A_+\mathbf{K}_+} e^{\ln(A_0)\mathbf{K}_0} e^{C_+\mathbf{K}_+} \\ &\quad \times e^{\ln(C_0)\mathbf{K}_0} e^{C_-\mathbf{K}_-},\end{aligned} \tag{7}$$

with $C_\pm$ and C_0 given by [14]:

$$
\begin{aligned}
C_+ &= \frac{B_0 B_+}{1 - B_0 B_+ (A_- + B_-)}, \\
C_- &= \frac{B_0 (A_- + B_-)}{1 - B_0 B_+ (A_- + B_-)}, \\
C_0 &= \frac{B_0}{[1 - B_0 B_+ (A_- + B_-)]^2}.
\end{aligned}
$$

The second step consists in using the relationship

$$e^{\ln(A_0)\mathbf{K}_0} e^{C_+ \mathbf{K}_+} = e^{A_0 C_+ \mathbf{K}_+} e^{\ln(A_0)\mathbf{K}_0}$$

for the second and third exponentials in the second equality on the rhs of equation (7), with the aim of establishing the intermediate result

$$\mathbf{T}(\Omega_\pm, \Omega_0)\mathbf{T}(\Lambda_\pm, \Lambda_0) = e^{(A_+ + A_0 C_+)\mathbf{K}_+} e^{\ln(A_0 C_0)\mathbf{K}_0} e^{C_- \mathbf{K}_-}.$$

The rhs of this equation represents the normal-order decomposition of an abstract operator defined as

$$\mathbf{T}(\Sigma_\pm, \Sigma_0) := e^{\Sigma_+ \mathbf{K}_+ + \Sigma_0 \mathbf{K}_0 + \Sigma_- \mathbf{K}_-},$$

where the c-number parameters $\Sigma_\pm$ and Σ_0 satisfy the following mathematical relations:

$$
\begin{aligned}
A_+ + A_0 C_+ &= \frac{(\Sigma_+/\beta)\sinh(\beta)}{\cosh(\beta) - (\Sigma_0/2\beta)\sinh(\beta)}, \\
C_- &= \frac{(\Sigma_-/\beta)\sinh(\beta)}{\cosh(\beta) - (\Sigma_0/2\beta)\sinh(\beta)}, \\
A_0 C_0 &= [\cosh(\beta) - (\Sigma_0/2\beta)\sinh(\beta)]^{-2},
\end{aligned}
$$

with $\beta = [(\Sigma_0/2)^2 - \Sigma_+ \Sigma_-]^{1/2}$. Consequently, substituting the definitions of $C_\pm$ and C_0 in these relations, we obtain a coupled set of nonlinear equations that permits us not only to establish a link between $\{\Sigma_\pm, \Sigma_0\}$ and $\{\Omega_\pm, \Omega_0, \Lambda_\pm, \Lambda_0\}$, but also to verify the general composition formula (2). ■

An interesting consequence from this mathematical statement is associated with the construction process of a general family of unitary transformations where the abstract operator (1) has a central role. To carry out this task let us first establish a corollary which is directly related to the Lemma II.1.

Corollary II.1 *For $\Omega_+ = \xi$, $\Omega_- = -\xi^*$, and $\Omega_0 = \mathrm{i}\omega$, with $\xi \in \mathbb{C}$ and $\omega \in \mathbb{R}$, the abstract operator (1) represents a generator of unitary transformations associated with the $\mathfrak{su}(1,1)$ Lie algebra.*

Proof II.2 *Basically, the idea is obtaining a specific subset of arbitrary c-number parameters such that*

$$\mathbf{T}(\Omega_\pm, \Omega_0)[\mathbf{T}(\Omega_\pm, \Omega_0)]^\dagger = [\mathbf{T}(\Omega_\pm, \Omega_0)]^\dagger \mathbf{T}(\Omega_\pm, \Omega_0) = \mathbf{1}.$$

Hence, let us initially investigate under what circumstances the mathematical relation

$$\mathbf{T}(\Omega_\pm, \Omega_0)[\mathbf{T}(\Omega_\pm, \Omega_0)]^\dagger = \mathbf{1}$$

is verified. In fact, this condition can be promptly derived from the general composition formula (2) for $\Lambda_\pm = \Omega_\mp^*$, $\Lambda_0 = \Omega_0^*$, and $\Sigma_\pm = \Sigma_0 = 0$, with the additional restrictions $\Lambda_\pm = -\Omega_\pm$ and $\Lambda_0 = -\Omega_0$. Consequently, the equalities $\Omega_\pm = -\Omega_\mp^*$ and $\Omega_0 = -\Omega_0^*$ are satisfied only for $\Omega_+ = \xi$, $\Omega_- = -\xi^*$, and $\Omega_0 = \mathrm{i}\omega$, with $\xi \in \mathbb{C}$ and $\omega \in \mathbb{R}$. This same particular subset of arbitrary c-number parameters can also be obtained from the analysis of

$$[\mathbf{T}(\Omega_\pm, \Omega_0)]^\dagger \mathbf{T}(\Omega_\pm, \Omega_0) = \mathbf{1},$$

which implies that $\mathbf{T}(\xi, \omega)$ gives a family of unitary transformations for a general class of representations associated with the $\mathfrak{su}(1,1)$ Lie algebra. ■

To illustrate our results, let us consider as a first example the single-mode bosonic representation of the Heisenberg-Weyl algebra where the generators $\mathbf{K}_\pm$ and $\mathbf{K}_0$ are expressed as $\mathbf{K}_+ = (1/2)\mathbf{a}^{\dagger 2}$, $\mathbf{K}_- = (1/2)\mathbf{a}^2$, and $\mathbf{K}_0 = (1/2)(\mathbf{a}^\dagger \mathbf{a} + 1/2)$, with $\mathbf{a}$ and $\mathbf{a}^\dagger$ being, respectively, the boson annihilation and creation operators satisfying the well-known commutation relation $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1}$. In this case, the unitary operator $\mathbf{T}(\xi, \omega)$ assumes the form

$$\mathbf{T}_1(\xi, \omega) = e^{\frac{1}{2}[\xi \mathbf{a}^{\dagger 2} + \mathrm{i}\omega(\mathbf{a}^\dagger \mathbf{a} + 1/2) - \xi^* \mathbf{a}^2]}, \tag{8}$$

which coincides with the squeezing operator

$$\mathbf{S}(\xi) := e^{\frac{1}{2}(\xi \mathbf{a}^{\dagger 2} - \xi^* \mathbf{a}^2)} \quad \text{when} \quad \omega = 0.$$

It is worth mentioning that, in particular, equation (8) can be promptly used to derive the quantum analogous of the Fresnel transform in classical optics [15].

Another interesting example encompasses the two-mode bosonic representation in which the generators are specifically given by $\mathbf{K}_+ = \mathbf{a}^\dagger \mathbf{b}^\dagger$, $\mathbf{K}_- = \mathbf{ab}$, and $\mathbf{K}_0 = (1/2)(\mathbf{a}^\dagger \mathbf{a} + \mathbf{b}^\dagger \mathbf{b} + 1)$, where $[\mathbf{a}, \mathbf{a}^\dagger] = [\mathbf{b}, \mathbf{b}^\dagger] = \mathbf{1}$. In this context, the unitary operator

$$\mathbf{T}_2(\xi, \omega) = e^{\xi \mathbf{a}^\dagger \mathbf{b}^\dagger + \mathrm{i}(\omega/2)(\mathbf{a}^\dagger \mathbf{a} + \mathbf{b}^\dagger \mathbf{b} + 1) - \xi^* \mathbf{ab}} \tag{9}$$

recovers the two-mode squeezing operator

$$\mathbf{S}_2(\xi) = \exp(\xi \mathbf{a}^\dagger \mathbf{b}^\dagger - \xi^* \mathbf{ab}) \quad \text{for} \quad \omega = 0.$$

Moreover, the action of $\mathbf{T}_2(\xi, \omega)$ in the annihilation operators for each mode of the electromagnetic field implies in the following results:

$$
\begin{aligned}
\mathbf{T}_2^\dagger(\xi, \omega)\mathbf{a}\mathbf{T}_2(\xi, \omega) &= [\cosh(\phi) + \mathrm{i}(\omega/2\phi)\sinh(\phi)]\,\mathbf{a} \\
&\quad + (\xi/\phi)\sinh(\phi)\mathbf{b}^\dagger, \qquad (10) \\
\mathbf{T}_2^\dagger(\xi, \omega)\mathbf{b}\mathbf{T}_2(\xi, \omega) &= [\cosh(\phi) + \mathrm{i}(\omega/2\phi)\sinh(\phi)]\,\mathbf{b} \\
&\quad + (\xi/\phi)\sinh(\phi)\mathbf{a}^\dagger, \qquad (11)
\end{aligned}
$$

being $\phi = [|\xi|^2 - (\omega/2)^2]^{1/2}$. This set of unitary transformations for $\omega = 0$ has its counterpart in the quantum description of physical processes involving parametric amplification [10, 16]. Recently, Pielawa *et al.* [17] have proposed a new method for generating two-mode squeezing

in high-Q resonators using a beam of atoms (which acts as a reservoir for the field) with random arrival times. In particular, the authors have used the unitary two-mode squeezing operator $\mathbf{S}_2(\xi)$ to bring an effective Hamiltonian – which describes resonant single-photon processes – to the well-known Jaynes-Cummings form, where the new bosonic operators are connected to the old ones by two-mode squeezing transformations. Another interesting application is based on the analogy between phonons in an axially time-dependent ion trap and quantum fields in an expanding/contracting universe, where the multimode squeezing operator represents the basic mechanism for cosmological particle creation [18].

It is worth noticing that there are some textbooks on quantum optics [3] which discuss particular cases (if one compares with those exposed here) of one- and two-mode squeezing operators in different contexts in physics and their connections with nonclassical states of the electromagnetic field. However, the dynamical origin of the real parameter ω present in equation (9), for example, has not been investigated up until now in literature, and this fact will be our object of study in the next section. For this intent, we obtain the exact solutions of the Heisenberg equations for a specific nonresonant system like the parametric amplifier, and show the link between time-evolution operator and $\mathbf{T}_2(\xi,\omega)$.

III. PARAMETRIC AMPLIFIER

The parametric amplifier model proposed by Louisell *et al.* [10] consists basically of two coupled modes of the electromagnetic field, which play a symmetrical role in the amplification process [16]. Such dynamical elements are usually described by the Hamiltonian ($\hbar = 1$)

$$\mathbf{H}(t) = \omega_a \mathbf{a}^{\dagger}\mathbf{a} + \omega_b \mathbf{b}^{\dagger}\mathbf{b} + \kappa\, \mathbf{a}\mathbf{b}\, e^{\mathrm{i}\eta t} + \kappa^{*} \mathbf{a}^{\dagger}\mathbf{b}^{\dagger} e^{-\mathrm{i}\eta t}, \quad (12)$$

where $\mathbf{a}$ ($\mathbf{b}$) is the annihilation operator for the signal (idler) mode, η provides the frequency of the pump field (which has been assumed strong enough to be expressed in classical terms), and κ represents a complex coupling constant being proportional to the second-order susceptibility of the nonlinear medium and to the amplitude of the pump. Moreover, let us introduce a small deviation δ in the usual definition of η such that $\eta = \omega_a + \omega_b + \delta$ with $\delta/(\omega_a + \omega_b) \ll 1$ (by hypothesis we are assuming that δ comes from a non-perfect match between the frequencies η and $\omega_a + \omega_b$). In this case, the solutions of the Heisenberg equations are given by

$$\begin{aligned} \mathbf{a}(t) &= e^{-\mathrm{i}(\omega_a+\delta/2)t}\Big\{ [\cosh(\varphi t) + \mathrm{i}(\delta/2\varphi)\sinh(\varphi t)]\,\mathbf{a}(0) \\ &\quad -\mathrm{i}(\kappa^{*}/\varphi)\sinh(\varphi t)\mathbf{b}^{\dagger}(0)\Big\}, \\ \mathbf{b}(t) &= e^{-\mathrm{i}(\omega_b+\delta/2)t}\Big\{ [\cosh(\varphi t) + \mathrm{i}(\delta/2\varphi)\sinh(\varphi t)]\,\mathbf{b}(0) \\ &\quad -\mathrm{i}(\kappa^{*}/\varphi)\sinh(\varphi t)\mathbf{a}^{\dagger}(0)\Big\}, \end{aligned}$$

plus their Hermitian conjugates for $\varphi = [|\kappa|^2 - (\delta/2)^2]^{1/2}$ fixed. These solutions lead us to verify that $[\mathbf{a}(t), \mathbf{a}^{\dagger}(t)] = [\mathbf{b}(t), \mathbf{b}^{\dagger}(t)] = \mathbf{1}$, which implies in the unitariety of the time-evolution operator $\mathbf{U}(t)$. This fact leads us to show that $\mathbf{n}_a(t) - \mathbf{n}_b(t) = \mathbf{n}_a(0) - \mathbf{n}_b(0)$ (conservation law) for $\mathbf{n}_c(t) := \mathbf{c}^{\dagger}(t)\mathbf{c}(t)$ is easily seen to hold; therefore, the intensity correlation function $\langle \mathbf{n}_a(t)\mathbf{n}_b(t)\rangle$ can be written as $\langle \mathbf{n}_a(t)\mathbf{n}_b(t)\rangle = \langle \mathbf{n}_a^2(t)\rangle + \langle \mathbf{n}_a(t)[\mathbf{n}_b(0) - \mathbf{n}_a(0)]\rangle$. For instance, when the initial state coincides with the number states $\{|n_a, n_b\rangle\}_{n_a,n_b\in\mathbb{N}}$, the second term on the rhs of this equation is reduced to $(n_b - n_a)\langle n_a, n_b|\mathbf{n}_a(t)|n_a, n_b\rangle$ for $n_a \neq n_b$. So, if one considers $n_a = n_b$, there is no contribution from this term and consequently, $\langle \mathbf{n}_a(t)\mathbf{n}_b(t)\rangle$ corresponds to the maximum violation of the Cauchy-Schwarz inequality $\langle \mathbf{a}^{\dagger}\mathbf{a}\mathbf{b}^{\dagger}\mathbf{b}\rangle \leq \langle \mathbf{a}^{\dagger 2}\mathbf{a}^2\rangle$ for the parametric amplifier [3].

Next, we show how the time-evolution operator $\mathbf{U}(t)$ can be connected with the unitary operator $\mathbf{T}_2(\xi,\omega)$. To carry out this task, let us initially mention that the time-dependent global phase factors present in the solutions of the Heisenberg equations are obtained through the action of the rotation operator

$$\mathbf{R}(\omega_a, \omega_b, \delta; t) = e^{-\mathrm{i}t\left[(\omega_a+\delta/2)\mathbf{a}^{\dagger}\mathbf{a} + (\omega_b+\delta/2)\mathbf{b}^{\dagger}\mathbf{b}\right]} \quad (13)$$

on the annihilation operators $\mathbf{a}(0)$ and $\mathbf{b}(0)$. The next step then consists in noticing that for $\phi = \varphi t$, $\omega = \delta t$, and $\xi = -\mathrm{i}\kappa^{*} t$, the operator

$$\mathbf{T}_2(\kappa, \delta; t) = e^{-\mathrm{i}t\left[\kappa^{*}\mathbf{a}^{\dagger}\mathbf{b}^{\dagger} - (\delta/2)(\mathbf{a}^{\dagger}\mathbf{a} + \mathbf{b}^{\dagger}\mathbf{b} + 1) + \kappa\,\mathbf{a}\mathbf{b}\right]} \quad (14)$$

is responsible for generating the terms between braces in the solutions $\mathbf{a}(t)$ and $\mathbf{b}(t)$ – see equations (10) and (11). After these considerations, it is easy to show that

$$\mathbf{U}(t) := \mathbf{R}(\omega_a, \omega_b, \delta; t)\mathbf{T}_2(\kappa, \delta; t) \quad (15)$$

provides the solutions obtained above by means of the mathematical operation sketched in the identities $\mathbf{a}(t) = \mathbf{U}^{\dagger}(t)\mathbf{a}(0)\mathbf{U}(t)$ and $\mathbf{b}(t) = \mathbf{U}^{\dagger}(t)\mathbf{b}(0)\mathbf{U}(t)$. Note that the particular nonresonant system here studied represents a first dynamical application of the results obtained in Section II for the two-mode bosonic representation, where the connection between the time-evolution operator (15) and the unitary operator (9) is promptly established. It is worth emphasizing that different physical systems can also be used for explaining the dynamical origin of the real parameter ω present in $\mathbf{T}_2(\xi,\omega)$ (e.g., see Ref. [17]), but this fact still deserves be carefully investigated.

IV. AN INITIAL STUDY ON ENTANGLEMENT VIA WIGNER FUNCTION

Let us initially consider a specific subset of bipartite physical systems described by continuous variables such that the parametric amplifier model here studied (or the simple model for parametric frequency conversion proposed by Louisell *et al.* [10], and subsequently studied in

detail by Tucker and Walls [19]) constitutes a particular element. Furthermore, let us also assume that $\boldsymbol{\rho}(t) = \mathbf{U}(t)\boldsymbol{\rho}(0)\mathbf{U}^{\dagger}(t)$ describes the dynamics of any bipartite system belonging to this subset whose unitary time-evolution operator $\mathbf{U}(t)$ is related to the Hamiltonian operator $\mathbf{H}(t)$; by hypothesis, the initial density operator $\boldsymbol{\rho}(0)$ represents the system prepared at the initial instant $t = 0$ in any disentangled (entangled) state. The symmetric characteristic function for this class of bipartite physical systems is given by $\mathscr{C}(\mathbb{G};t) := \mathrm{Tr}[\mathbf{D}(\mathbb{G})\boldsymbol{\rho}(t)]$, where $\mathbf{D}(\mathbb{G}) := \exp(-\mathbb{G}^{\dagger}\mathbb{E}\mathbb{O})$ defines a displacement operator written in terms of the matrices $\mathbb{G}^{\dagger} = \begin{pmatrix} \xi_a^* & \xi_a & \xi_b^* & \xi_b \end{pmatrix}$, $\mathbb{E} = \mathbb{I} \otimes \mathbb{S}$ with $\mathbb{I} = \mathrm{diag}(1,1)$ and $\mathbb{S} = \mathrm{diag}(1,-1)$ being the respective 2×2 identity and symplectic matrices, and $\mathbb{O}^{\dagger} = \begin{pmatrix} \mathbf{a}^{\dagger} & \mathbf{a} & \mathbf{b}^{\dagger} & \mathbf{b} \end{pmatrix}$. Note that due to the cyclic property of the trace operation, the symmetric characteristic function can also be evaluated through the mathematical statement $\mathrm{Tr}[\mathbf{U}^{\dagger}(t)\mathbf{D}(\mathbb{G})\mathbf{U}(t)\boldsymbol{\rho}(0)]$, where the time evolution of the displacement operator plays an important role.

Next, let us suppose that the action of the unitary time-evolution operator on the matrix $\mathbb{O}$ transforms the annihilation (creation) operators $\mathbf{a}$ ($\mathbf{a}^{\dagger}$) and $\mathbf{b}$ ($\mathbf{b}^{\dagger}$) following the general rule $\mathbb{O}_{\mathrm{H}}(t) = \mathbf{U}^{\dagger}(t)\mathbb{O}\mathbf{U}(t) = \mathbb{T}(t)\mathbb{O}$, where

$$\mathbb{T}(t) = \begin{bmatrix} \mu_a(t) & \nu_a(t) & \chi_a(t) & \eta_a(t) \\ \nu_a^*(t) & \mu_a^*(t) & \eta_a^*(t) & \chi_a^*(t) \\ \mu_b(t) & \nu_b(t) & \chi_b(t) & \eta_b(t) \\ \nu_b^*(t) & \mu_b^*(t) & \eta_b^*(t) & \chi_b^*(t) \end{bmatrix}$$

represents a 4×4 matrix whose elements are c-number functions determined from the Heisenberg equations for $\mathbb{O}_{\mathrm{H}}(t)$ with specific initial conditions that preserve the quantum mechanics (the subscript H indicates that operators are in the Heisenberg picture). For instance, when $t = 0$ the c-number functions should necessarily imply in the mathematical identity $\mathbb{T}(0) = \mathrm{diag}(1,1,1,1)$; while for $t \geq 0$ the commutation relations

$$\begin{aligned} \left[\mathbf{a}_{\mathrm{H}}(t), \mathbf{a}_{\mathrm{H}}^{\dagger}(t)\right] &= \mathbf{1}, \quad \left[\mathbf{b}_{\mathrm{H}}(t), \mathbf{b}_{\mathrm{H}}^{\dagger}(t)\right] = \mathbf{1}, \\ \left[\mathbf{a}_{\mathrm{H}}(t), \mathbf{b}_{\mathrm{H}}(t)\right] &= 0, \quad \left[\mathbf{a}_{\mathrm{H}}(t), \mathbf{b}_{\mathrm{H}}^{\dagger}(t)\right] = 0, \end{aligned}$$

provide extra relations for the c-number functions which permit us to solve completely the Heisenberg equations, namely

$$\begin{aligned} |\mu_a(t)|^2 - |\nu_a(t)|^2 + |\chi_a(t)|^2 - |\eta_a(t)|^2 &= 1, \\ |\mu_b(t)|^2 - |\nu_b(t)|^2 + |\chi_b(t)|^2 - |\eta_b(t)|^2 &= 1, \\ \mu_a(t)\nu_b(t) - \nu_a(t)\mu_b(t) + \chi_a(t)\eta_b(t) - \eta_a(t)\chi_b(t) &= 0, \\ \mu_a(t)\mu_b^*(t) - \nu_a(t)\nu_b^*(t) + \chi_a(t)\chi_b^*(t) - \eta_a(t)\eta_b^*(t) &= 0. \end{aligned}$$

The first immediate consequence of these results is that $\mathbf{U}^{\dagger}(t)\mathbf{D}(\mathbb{G})\mathbf{U}(t)$ will produce a new displacement operator $\mathbf{D}(\mathbb{Y}) = \exp(-\mathbb{Y}^{\dagger}\mathbb{E}\mathbb{O})$ with $\mathbb{Y}^{\dagger} = \begin{pmatrix} \beta_a^* & \beta_a & \beta_b^* & \beta_b \end{pmatrix}$, where the new elements are connected with the old ones through the equalities

$$\begin{aligned} \beta_a &= \mu_a^*(t)\xi_a - \nu_a(t)\xi_a^* + \mu_b^*(t)\xi_b - \nu_b(t)\xi_b^*, \\ \beta_b &= \chi_a^*(t)\xi_a - \eta_a(t)\xi_a^* + \chi_b^*(t)\xi_b - \eta_b(t)\xi_b^*, \end{aligned}$$

plus their respective complex conjugates. For convenience in our calculations, $\beta_{a(b)}$ means a short notation for $\beta_{a(b)} = \beta_{a(b)}(\xi_a, \xi_b; t)$. It is worth noticing that $\mathscr{C}(\mathbb{G};t)$ may now be written as $\mathscr{C}(\mathbb{Y};0) = \mathrm{Tr}[\mathbf{D}(\mathbb{Y})\boldsymbol{\rho}(0)]$ (i.e., the symmetric characteristic function $\mathscr{C}(\mathbb{G};t)$ is thus specified in terms of the form it takes at $t = 0$, which corroborates the results obtained by Mollow and Glauber [16] for the parametric amplifier model), and this fact will bring some insights into the study of entanglement for bipartite systems via Wigner function.

The Wigner function for this particular subset of bipartite physical systems may then be defined as the four-dimensional Fourier transform of the symmetric characteristic function $\mathscr{C}(\mathbb{G};t)$, that is,

$$\mathscr{W}(\mathbb{X};t) = \int \frac{d^2\xi_a d^2\xi_b}{\pi^2} \exp\left(\mathbb{G}^{\dagger}\mathbb{E}\mathbb{X}\right) \mathscr{C}(\mathbb{G};t), \tag{16}$$

with $\mathbb{X}^{\dagger} := \begin{pmatrix} \alpha_a^* & \alpha_a & \alpha_b^* & \alpha_b \end{pmatrix}$. Note that the variables of integration ξ_a and ξ_b can be changed to β_a and β_b in this equation, once the Jacobian matrix of the transformation has a determinant whose absolute value is equal to one – this fact implies in the identity $d^2\xi_a d^2\xi_b = d^2\beta_a d^2\beta_b$. Hence, the integration can now be conveniently carried out through the mathematical relation

$$\begin{aligned} \mathscr{W}(\mathbb{X};t) &= \int \frac{d^2\beta_a d^2\beta_b}{\pi^2} \exp\left(\mathbb{Y}^{\dagger}\mathbb{E}\mathbb{Z}\right) \mathscr{C}(\mathbb{Y};0) \\ &= \mathscr{W}(\mathbb{Z};0), \end{aligned} \tag{17}$$

where $\mathbb{Z}^{\dagger} := \begin{pmatrix} \gamma_a^* & \gamma_a & \gamma_b^* & \gamma_b \end{pmatrix}$ defines a new matrix with elements given by[30]

$$\begin{aligned} \gamma_a &= \mu_a^*(t)\alpha_a - \nu_a(t)\alpha_a^* + \mu_b^*(t)\alpha_b - \nu_b(t)\alpha_b^*, \\ \gamma_b &= \chi_a^*(t)\alpha_a - \eta_a(t)\alpha_a^* + \chi_b^*(t)\alpha_b - \eta_b(t)\alpha_b^*, \end{aligned}$$

and their respective complex conjugates. Thus, equation (17) asserts that $\mathscr{W}(\mathbb{X};t)$ can also be expressed in terms of the form it takes at the initial time $t = 0$, since the new variables $\gamma_a(\alpha_a, \alpha_b; t)$ and $\gamma_b(\alpha_a, \alpha_b; t)$ carry the information of the *dynamical entanglement* between the variables α_a and α_b. Furthermore, this result permits us to show that for a given bipartite system initially prepared in any entangled state, it will remain entangled for all $t \geq 0$; otherwise, if the density operator is described at $t = 0$ as $\boldsymbol{\rho}(0) = \boldsymbol{\rho}_a(0) \otimes \boldsymbol{\rho}_b(0)$ (disentangled state), the appearance of entanglement in the Wigner function will depend exclusively on the dynamics provided by the Hamiltonian operator $\mathbf{H}(t)$[31] – here associated with the time-dependent matrix $\mathbb{Z}$. Following, let us apply the results obtained until now to the parametric amplifier model, where different initial states of the electromagnetic field will be considered.

Now we focus our efforts in evaluating $\mathscr{W}(\mathbb{Z};0)$ for a well-known family of two-mode electromagnetic fields where, in particular, both modes are initially prepared in the coherent states, number states, and thermal states, respectively. For instance, the symmetric characteristic

functions in these situations are given by

$$\begin{aligned}\mathscr{C}_{\rm coh}(\mathbb{Y};0) &= e^{-\frac{1}{2}\left(|\beta_a|^2+|\beta_b|^2\right)+2{\rm i}[{\rm Im}(\beta_a\zeta_a^*)+{\rm Im}(\beta_b\zeta_b^*)]},\\ \mathscr{C}_{\rm n}(\mathbb{Y};0) &= e^{-\frac{1}{2}\left(|\beta_a|^2+|\beta_b|^2\right)}L_{n_a}(|\beta_a|^2)L_{n_b}(|\beta_b|^2),\\ \mathscr{C}_{\rm th}(\mathbb{Y};0) &= e^{-\frac{1}{2}\left[(1+2\bar{n}_a)|\beta_a|^2+(1+2\bar{n}_b)|\beta_b|^2\right]},\end{aligned}$$

while their respective Wigner functions can be expressed as

$$\mathscr{W}_{\rm coh}(\mathbb{Z};0) = 4\,e^{-2\left(|\gamma_a-\zeta_a|^2+|\gamma_b-\zeta_b|^2\right)}, \qquad (18)$$

$$\begin{aligned}\mathscr{W}_{\rm n}(\mathbb{Z};0) &= 4(-1)^{n_a+n_b}e^{-2\left(|\gamma_a|^2+|\gamma_b|^2\right)}L_{n_a}(4|\gamma_a|^2)\\ &\quad\times L_{n_b}(4|\gamma_b|^2),\end{aligned} \qquad (19)$$

$$\begin{aligned}\mathscr{W}_{\rm th}(\mathbb{Z};0) &= 4\left[(1+2\bar{n}_a)(1+2\bar{n}_b)\right]^{-1}\\ &\quad\times e^{-2\left[(1+2\bar{n}_a)^{-1}|\gamma_a|^2+(1+2\bar{n}_b)^{-1}|\gamma_b|^2\right]},\end{aligned} \qquad (20)$$

with $L_n(z)$ denoting a Laguerre polynomial and $\bar{n}_{a(b)}$ being the mean number of photons for a chaotic light field. Note that if we consider the solutions obtained from the Heisenberg equations for the parametric amplifier model (see previous section), it is easy to show that $\mu_a(t)$, $\eta_a(t)$, $\nu_b(t)$, and $\chi_b(t)$ (once the further time-dependent coefficients do no exist) determine completely the c-numbers $\beta_{a(b)}(\xi_a,\xi_b;t)$ and $\gamma_{a(b)}(\alpha_a,\alpha_b;t)$. Indeed, this last step leads us to characterize precisely $\mathscr{C}(\mathbb{Y};0)$ and $\mathscr{W}(\mathbb{Z};0)$. Moreover, when $\delta=0$, $\kappa\in\mathbb{R}_+$, and $(\omega_a+\omega_b)t=3\pi/2$, Eq. (18) coincides with $\mathscr{W}_{\rm EPR}(\gamma_a,\gamma_b)$ for $\zeta_{a(b)}=0$ (two-mode vacuum state), this function being that used by Braunstein and Kimble [22] in the theoretical description of teleportation involving continuous quantum variables.[32]

Finally, let us say some few words about the results obtained in this section. It is important to emphasize that equation (17) permits us to describe qualitatively the entanglement of a specific subset of bipartite systems, where now we can promptly identify two distinct origins of this quantum effect: the first associated with the entanglement in the initial conditions (here labelled by kinematical entanglement), while the second is responsible for the dynamical entanglement via Hamiltonian operator. In addition, the normally ordered moments

$$\begin{aligned}\langle \mathbf{a}^{\dagger p}(t)\mathbf{a}^{q}(t)\mathbf{b}^{\dagger r}(t)\mathbf{b}^{s}(t)\rangle &= \mathbf{\Gamma}^{(p,q,r,s)}_{\xi_{\rm a},\xi_{\rm a}^*,\xi_{\rm b},\xi_{\rm b}^*}e^{\frac{1}{2}\left(|\xi_{\rm a}|^2+|\xi_{\rm b}|^2\right)}\\ &\quad\times\mathscr{C}(\xi_a,\xi_a^*,\xi_b,\xi_b^*;t)\Big|_{\xi_{\rm a},\xi_{\rm a}^*,\xi_{\rm b},\xi_{\rm b}^*=0},\end{aligned} \qquad (21)$$

with

$$\mathbf{\Gamma}^{(p,q,r,s)}_{\xi_{\rm a},\xi_{\rm a}^*,\xi_{\rm b},\xi_{\rm b}^*} := (-1)^{q+s}\frac{\partial^{p+q+r+s}}{\partial\xi_{\rm a}^p\partial\xi_{\rm a}^{*q}\partial\xi_{\rm b}^r\partial\xi_{\rm b}^{*s}}$$

and $\{p,q,r,s\}\in\mathbb{N}$, can also be used to investigate some recent proposals of inseparability criteria for continuous bipartite quantum states [21, 23]. In the next section, we will establish a reasonable measure of entanglement which is based on the results obtained by Piątek and Leoński [11] for the intermode correlations in phase space.

V. ENTROPY FUNCTIONALS FOR CONTINUOUS PHASE-SPACE REPRESENTATIONS

In order to establish a quantitative estimate of entanglement for bipartite systems, let us introduce some basic definitions of entropy functionals in continuous phase-space representations. The first definition is based on the joint entropy [11]

$${\rm E}[\mathscr{H};t] := -\int\frac{d^2\alpha_a d^2\alpha_b}{\pi^2}\,\mathscr{H}(\alpha_a,\alpha_b;t)\ln\left[\mathscr{H}(\alpha_a,\alpha_b;t)\right], \qquad (22)$$

where $\mathscr{H}(\alpha_a,\alpha_b;t) := \langle\alpha_a,\alpha_b|\boldsymbol{\rho}(t)|\alpha_a,\alpha_b\rangle$ denotes the Husimi function in the continuous coherent-state representations for a bipartite system described by the density operator $\boldsymbol{\rho}(t)$. Note that ${\rm E}[\mathscr{H};t]$ presents certain properties inherent to its definition which deserve be mentioned: (i) the probability distribution function $\mathscr{H}(\alpha_a,\alpha_b;t)$ is strictly positive and limited to the interval $[0,1]$; consequently, the joint entropy (22) characterizes a well-defined function whose behaviour does not present any mathematical inconsistencies. Moreover, (ii) this definition essentially measures the functional correlation between the continuous variables used to describe each part of the joint system. Hence, ${\rm E}[\mathscr{H};t]$ can be considered as a natural extension of that definition employed by Wehrl [12] for information entropy.[33]

The second definition consists of functionals related to the partial entropies

$${\rm E}[\mathscr{H}^{(\rm A)};t] := -\int\frac{d^2\alpha_a}{\pi}\,\mathscr{H}^{(\rm A)}(\alpha_a;t)\ln\left[\mathscr{H}^{(\rm A)}(\alpha_a;t)\right] \qquad (23)$$

and

$${\rm E}[\mathscr{H}^{(\rm B)};t] := -\int\frac{d^2\alpha_b}{\pi}\,\mathscr{H}^{(\rm B)}(\alpha_b;t)\ln\left[\mathscr{H}^{(\rm B)}(\alpha_b;t)\right], \qquad (24)$$

which depend basically on the marginal Husimi functions

$$\begin{aligned}\mathscr{H}^{(\rm A)}(\alpha_a;t) &= \int\frac{d^2\alpha_b}{\pi}\mathscr{H}(\alpha_a,\alpha_b;t),\\ \mathscr{H}^{(\rm B)}(\alpha_b;t) &= \int\frac{d^2\alpha_a}{\pi}\mathscr{H}(\alpha_a,\alpha_b;t).\end{aligned}$$

Such marginal Husimi functions carry information of the entanglement between the subsystems 'A' and 'B', since the partial trace over the continuous variables $\alpha_{a(b)}$ of a particular subsystem 'A' ('B') allows the introduction, via time-evolution operator $\mathbf{U}(t)$ and/or initial density operator $\boldsymbol{\rho}(0)$, of important correlations in the bipartite states.

Let us derive now some mathematical relations among these entropy functionals from the Araki-Lieb inequality [25], i.e.,

$$\begin{aligned}\left|{\rm E}[\mathscr{H}^{(\rm A)};t]-{\rm E}[\mathscr{H}^{(\rm B)};t]\right| &\leq {\rm E}[\mathscr{H};t]\leq{\rm E}[\mathscr{H}^{(\rm A)};t]\\ &\quad+ {\rm E}[\mathscr{H}^{(\rm B)};t].\end{aligned} \qquad (25)$$

For instance, the rhs of this inequality corresponds to the subadditivity property for the Wehrl's entropy functionals, while the equal sign reflects the complete disentanglement between the subsystems 'A' and 'B' of the joint system. In addition, the conditional entropies [11, 12]

$$\mathrm{E}[\mathscr{H}/\mathscr{H}^{(\mathrm{A})};t] = \mathrm{E}[\mathscr{H};t] - \mathrm{E}[\mathscr{H}^{(\mathrm{A})};t] \tag{26}$$

and

$$\mathrm{E}[\mathscr{H}/\mathscr{H}^{(\mathrm{B})};t] = \mathrm{E}[\mathscr{H};t] - \mathrm{E}[\mathscr{H}^{(\mathrm{B})};t] \tag{27}$$

lead us not only to establish the balance equation

$$\mathrm{E}[\mathscr{H}/\mathscr{H}^{(\mathrm{A})};t]+\mathrm{E}[\mathscr{H}^{(\mathrm{A})};t] = \mathrm{E}[\mathscr{H}/\mathscr{H}^{(\mathrm{B})};t]+\mathrm{E}[\mathscr{H}^{(\mathrm{B})};t], \tag{28}$$

but also to determine the further inequalities

$$\mathrm{E}[\mathscr{H}/\mathscr{H}^{(\mathrm{A})};t] \leq \mathrm{E}[\mathscr{H}^{(\mathrm{B})};t]$$

and

$$\mathrm{E}[\mathscr{H}/\mathscr{H}^{(\mathrm{B})};t] \leq \mathrm{E}[\mathscr{H}^{(\mathrm{A})};t]$$

from the subadditivity property. It is worth noticing that the equal signs hold in both situations only when the c-numbers α_a and α_b are functionally uncorrelated, namely, the bipartite Husimi function $\mathscr{H}(\alpha_a,\alpha_b;t)$ factorizes in the product of the marginal Husimi functions $\mathscr{H}^{(\mathrm{A})}(\alpha_a;t)\mathscr{H}^{(\mathrm{B})}(\alpha_b;t)$.

In what concerns the class of entropic functionals listed until now, it is convenient to define a new functional limited to the closed interval $[0,1]$ for any $t \geq 0$, which permits us to avoid any ambiguity in the significance of the subadditivity property. Thus, let us introduce the correlation functional

$$\mathrm{C}[\mathscr{H};t] := 2\left(1 - \frac{\mathrm{E}[\mathscr{H};t]}{\mathrm{E}[\mathscr{H}^{(\mathrm{A})};t]+\mathrm{E}[\mathscr{H}^{(\mathrm{B})};t]}\right), \tag{29}$$

which can be used to measure the functional correlation (that is, the 'degree of entanglement') between the parts 'A' and 'B' of the joint system, having as reference a factorizable bipartite Husimi function $\mathscr{H}(\alpha_a,\alpha_b;t)$. In this definition, the inferior limit of the closed interval $[0,1]$ corresponds to the situation where the bipartite Husimi function factorizes, while its superior limit reflects the opposite situation (i.e., when the bipartite states are maximally correlated); besides, the global factor '2' is associated with the number of possible combinations between the constituent parts of the system under investigation. As a first practical application of these entropy functionals we will consider two different examples of initially uncoupled bipartite states for the parametric amplifier model.

A. Coherent states

In this first example, we consider the bipartite pure states $\boldsymbol{\rho}(0) = |\zeta_a,\zeta_b\rangle\langle\zeta_a,\zeta_b|$ as being the initial state at $t=0$ of the model governed by the Hamiltonian (12). Here, $|\zeta_a,\zeta_b\rangle = |\zeta_a\rangle \otimes |\zeta_b\rangle$ where $|\zeta_{a(b)}\rangle$ characterizes the coherent states related to the signal (idler) mode of the electromagnetic field. Thus, after some nontrivial algebra, the bipartite Husimi function assumes the closed-form

$$\begin{aligned}\mathscr{H}_{\mathrm{coh}}(\alpha_a,\alpha_b;t) &= |A_0| e^{-\left(|\gamma_a-\zeta_a|^2+|\gamma_b-\zeta_b|^2\right)} \\ &\quad\times e^{2\mathrm{Re}\left[A_+^*(\gamma_a-\zeta_a)(\gamma_b-\zeta_b)\right]}\end{aligned} \tag{30}$$

such that

$$\begin{aligned}\gamma_{a(b)} &= e^{\mathrm{i}(\omega_{a(b)}+\delta/2)t}\left[\cosh(\varphi t) - \mathrm{i}(\delta/2\varphi)\sinh(\varphi t)\right]\alpha_{a(b)} \\ &\quad + \mathrm{i}(\kappa^*/\varphi)e^{-\mathrm{i}(\omega_{b(a)}+\delta/2)t}\sinh(\varphi t)\,\alpha^*_{b(a)}, \\ A_+ &= \frac{\mathrm{i}(\kappa^*/\varphi)\sinh(\varphi t)}{\cosh(\varphi t)+\mathrm{i}(\delta/2\varphi)\sinh(\varphi t)}, \\ A_0 &= \left[\cosh(\varphi t)+\mathrm{i}(\delta/2\varphi)\sinh(\varphi t)\right]^{-2}.\end{aligned}$$

Furthermore, the marginal Husimi functions

$$\mathscr{H}^{(\mathrm{A})}_{\mathrm{coh}}(\alpha_a;t) = |A_0| e^{-|A_0||\alpha_a-\epsilon_a|^2} \tag{31}$$

and

$$\mathscr{H}^{(\mathrm{B})}_{\mathrm{coh}}(\alpha_b;t) = |A_0| e^{-|A_0||\alpha_b-\epsilon_b|^2}, \tag{32}$$

with

$$\begin{aligned}\epsilon_{a(b)} &= \Big\{\left[\cosh(\varphi t)+\mathrm{i}(\delta/2\varphi)\sinh(\varphi t)\right]\zeta_{a(b)} \\ &\quad -\mathrm{i}(\kappa^*/\varphi)\sinh(\varphi t)\zeta^*_{b(a)}\Big\} e^{-\mathrm{i}(\omega_{a(b)}+\delta/2)t},\end{aligned}$$

permit us to verify how the dynamical entanglement introduces correlations between the continuous variables $\zeta_{a(b)}$ and $\zeta^*_{b(a)}$ through the label $\epsilon_{a(b)}$. Indeed, for $t=0$ it is possible to demonstrate that $\mathscr{H}_{\mathrm{coh}}(\alpha_a,\alpha_b;0)$ factorizes in the product $\mathscr{H}^{(\mathrm{A})}_{\mathrm{coh}}(\alpha_a;0)\mathscr{H}^{(\mathrm{B})}_{\mathrm{coh}}(\alpha_b;0)$. However, if one considers $t>0$, the dynamics governed by the process of parametric amplification introduces significant correlations between the signal and idler modes. Following, let us quantify these correlations with the help of Eq. (29).

The Gaussian structures present in the bipartite and marginal Husimi functions allow us to obtain closed form expressions for the respective joint and partial entropies, namely,

$$\mathrm{E}[\mathscr{H}_{\mathrm{coh}};t] = 2 + \ln\left[1+(|\kappa|/\varphi)^2\sinh^2(\varphi t)\right] \tag{33}$$

and

$$\mathrm{E}[\mathscr{H}^{(\mathrm{A,B})}_{\mathrm{coh}};t] = 1 + \ln\left[1+(|\kappa|/\varphi)^2\sinh^2(\varphi t)\right]. \tag{34}$$

Consequently, the correlation functional can be promptly evaluated as follows:

$$\mathrm{C}[\mathscr{H}_{\mathrm{coh}};t] = \frac{\ln\left[1+(|\kappa|/\varphi)^2\sinh^2(\varphi t)\right]}{1+\ln\left[1+(|\kappa|/\varphi)^2\sinh^2(\varphi t)\right]}. \tag{35}$$

Besides, if one considers the limit $\varphi t \gg 1$ for sufficiently long time, the correlation functional goes asymptotically to 1, this value being associated with the specific dynamics given by the Hamiltonian (12). On the other hand, for $\varphi t = 0$ we obtain $\mathrm{C}[\mathscr{H}_{\mathrm{coh}}; 0] = 0$, which corroborates the factorization process of the bipartite Husimi function $\mathscr{H}_{\mathrm{coh}}(\alpha_a, \alpha_b; 0)$.

B. Thermal states

Now, let us suppose that both signal and idler modes were initially prepared in the thermal states. In this case, the initial density operator admits the expansion

$$\boldsymbol{\rho}_{\mathrm{th}}(0) = \frac{1}{\bar{n}_a \bar{n}_b} \int \frac{d^2\zeta_a d^2\zeta_b}{\pi^2} e^{-\left(\bar{n}_a^{-1}|\zeta_a|^2 + \bar{n}_b^{-1}|\zeta_b|^2\right)} \mathbf{P}(\zeta_a, \zeta_b),$$

where $\bar{n}_{a(b)}$ represents the average occupation number of each mode, and $\mathbf{P}(\zeta_a, \zeta_b) = |\zeta_a, \zeta_b\rangle\langle\zeta_a, \zeta_b|$ is the projector of the coherent states. This typical example of bipartite mixed states constitutes an important starting point in the investigation process of the dynamical entanglement due to the parametric amplifier model. So, after some calculations, the bipartite Husimi function can be written as

$$\begin{aligned} \mathscr{H}_{\mathrm{th}}(\alpha_a, \alpha_b; t) &= [\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,1;t)]^{-1} \, e^{-\left(\ell_b|\alpha_a|^2 + \ell_a|\alpha_b|^2\right)} \\ &\quad \times e^{2\mathrm{Re}(\ell_{ab}\alpha_a\alpha_b)} \end{aligned} \tag{36}$$

with

$$\begin{aligned} \ell_a &= \frac{\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,0;t)}{\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,1;t)}, \qquad \ell_b = \frac{\mathscr{G}_{\bar{n}_a,\bar{n}_b}(0,1;t)}{\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,1;t)}, \\ \ell_{ab} &= \mathrm{i}(\kappa/\varphi)\sinh(\varphi t)\left[\cosh(\varphi t) - \mathrm{i}(\delta/2\varphi)\sinh(\varphi t)\right] \\ &\quad \times (\bar{n}_a + \bar{n}_b + 1)\left[\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,1;t)\right]^{-1} e^{\mathrm{i}(\omega_a+\omega_b+\delta)t}, \end{aligned}$$

and

$$\begin{aligned} \mathscr{G}_{\bar{n}_a,\bar{n}_b}(x,y;t) &= (\bar{n}_a x + 1)(\bar{n}_b y + 1) \\ &\quad + (\bar{n}_a + \bar{n}_b + 1)(|\kappa|/\varphi)^2 \sinh^2(\varphi t); \end{aligned}$$

while its joint entropy assumes the closed-form

$$\mathrm{E}[\mathscr{H}_{\mathrm{th}}; t] = 2 + \ln\left[\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,1;t)\right]. \tag{37}$$

In addition, the marginal Husimi functions

$$\mathscr{H}_{\mathrm{th}}^{(\mathrm{A})}(\alpha_a; t) = [\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,0;t)]^{-1} \, e^{-\left[\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,0;t)\right]^{-1}|\alpha_a|^2} \tag{38}$$

and

$$\mathscr{H}_{\mathrm{th}}^{(\mathrm{B})}(\alpha_b; t) = [\mathscr{G}_{\bar{n}_a,\bar{n}_b}(0,1;t)]^{-1} \, e^{-\left[\mathscr{G}_{\bar{n}_a,\bar{n}_b}(0,1;t)\right]^{-1}|\alpha_b|^2} \tag{39}$$

permit also to determine explicitly the partial entropies

$$\mathrm{E}[\mathscr{H}_{\mathrm{th}}^{(\mathrm{A})}; t] = 1 + \ln\left[\mathscr{G}_{\bar{n}_a,\bar{n}_b}(1,0;t)\right], \tag{40}$$

$$\mathrm{E}[\mathscr{H}_{\mathrm{th}}^{(\mathrm{B})}; t] = 1 + \ln\left[\mathscr{G}_{\bar{n}_a,\bar{n}_b}(0,1;t)\right]. \tag{41}$$

In this moment becomes important noticing that the subadditivity property is not violated for any $t \geq 0$, the equality $\mathrm{E}[\mathscr{H}_{\mathrm{th}}; 0] = \mathrm{E}[\mathscr{H}_{\mathrm{th}}^{(\mathrm{A})}; 0] + \mathrm{E}[\mathscr{H}_{\mathrm{th}}^{(\mathrm{B})}; 0]$ being consistent with the factorization of $\mathscr{H}_{\mathrm{th}}(\alpha_a, \alpha_b; 0)$ in the product $\mathscr{H}_{\mathrm{th}}^{(\mathrm{A})}(\alpha_a; 0)\mathscr{H}_{\mathrm{th}}^{(\mathrm{B})}(\alpha_b; 0)$.

Following, let us investigate the correlation functional $\mathrm{C}[\mathscr{H}_{\mathrm{th}}; t]$ for $t > 0$, since $\mathrm{C}[\mathscr{H}_{\mathrm{th}}; 0] = 0$ reflects the disentanglement between the bipartite mixed states. In fact, for $t \neq 0$ this functional assumes any values into the open interval $(0, 1)$, the maximum value 1 being reached only for sufficiently long time. Hence, we can conclude that: (i) any initially uncoupled states become entangled in the process described by the Hamiltonian (12), which confirms the results obtained by Dodonov *et al.* [21] through the evaluation of the inverse negativity coefficient (this measure also leads us to estimate quantitatively the 'degree of entanglement' of a particular class of bipartite states such as those studied in this section); furthermore, (ii) the dynamics here characterized by the physical system under investigation produces a maximum estimate of entanglement for any initially disentangled states, this value (namely, 1) being considered as a specific quantum signature of the parametric amplifier model.

VI. CONCLUSIONS

We have established a set of interesting formal results within the scope of quantum optics and quantum information theory that allows us, among other things,

- to define a new class of unitary squeezing transformations related to $\mathfrak{su}(1,1)$ Lie algebra which generalizes, for certain particular representations of its generators, the one- and two-mode squeezing operators [3];

- to discuss, from the physical point of view, how the generalized two-mode squeezing operator can be generated through a slightly modified version of the parametric amplifier model [10];

- to obtain a general integral representation for the bipartite Wigner function whose integrand is expressed as a product of two terms which are responsible for the dynamical and kinematical entanglement; and finally,

- to estimate quantitatively the 'degree of entanglement' related to an ideal bipartite system (we are discarding the unvoidable coupling with the environment in this context) by means of a theoretical framework [11] which is based on the Wehrl's approach [12] for the entropy functionals.

In fact, equation (29) and its respective properties mentioned properly in the body of the text permit us to estimate, through an entropic approach, the entanglement

effects for a wide class of electromagnetic field states, including *Gaussian* and *non-Gaussian* states. As a concluding remark, it is worth mentioning that these results have also potential applications in modern research on quantum teleportation [22], quantum tomography [4, 26], and quantum computation [27] (once continuous-variable entanglement can also be efficiently produced using squeezed light and linear optics [28]), as well as on the foundations of quantum mechanics through its extensions to the $\mathfrak{su}(2)$ Lie algebra [29].

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[30] Note that the extra relations obtained from the commutation relations for the time-dependent c-number functions lead us to determine the inverse matrix $\mathbb{T}^{-1}(t)$ and consequently, to establish the identities $\mathbb{Y} = \mathbb{T}^{-1}(t)\mathbb{G}$ and $\mathbb{Z} = \mathbb{T}^{-1}(t)\mathbb{X}$.

[31] According to Mollow and Glauber [16]: 'The fact that the Wigner function has this property is a consequence of the form taken by the Hamiltonian (12). It may be shown that whenever the Hamiltonian of a system of oscillators is given by a quadratic form in the creation and annihilation operators, the Wigner function is constant along classical trajectories. This property does not extend to systems with arbitrary Hamiltonians, as it does in the case of the classical phase-space distribution.' Beyond these fundamental features, it is worth mentioning that recent studies on the characterization and quantification of entanglement for symmetric and asymmetric bipartite Gaussian states have contributed considerably to our comprehension of this important nonclassical effect in quantum optics and quantum information theory [20, 21]. In this sense, we believe that the formalism here presented for the Wigner function can help to extend such results in order to include some important dynamical effects that allow us to better understand certain peculiarities on the entanglement process for a specific subset of bipartite physical systems described by continuous variables [2].

[32] It is worth noticing that the cases here studied represent a specific set of disentangled bipartite states whose characteristic and Wigner functions are written as a product of two functions associated with each mode separately, i.e., since $\boldsymbol{\rho}(0) = \boldsymbol{\rho}_a(0) \otimes \boldsymbol{\rho}_b(0)$ we promptly obtain the relations $\mathscr{C}(\mathbb{Y};0) = \mathscr{C}_a(\beta_a;0)\mathscr{C}_b(\beta_b;0)$ and $\mathscr{W}(\mathbb{Z};0) = \mathscr{W}_a(\gamma_a;0)\mathscr{W}_b(\gamma_b;0)$. In this situation, the dynamical entanglement originated from the Hamiltonian $\mathbf{H}(t)$ allows us to correlate the complex variables α_a and α_b (or ξ_a and ξ_b) present in $\gamma_{a(b)}$ ($\beta_{a(b)}$).

[33] Recently, Mintert and Życzkowski [24] evaluated explicitly the Wehrl and generalized Rényi-Wehrl entropy functionals for any pure states describing $N \times N$ bipartite quantum systems. For this intent, they properly defined the Husimi function for the $\mathfrak{su}(N) \times \mathfrak{su}(N)$ coherent-state representations, and showed that: (i) the Wehrl entropy functional is minimal iff the pure states above mentioned are separable, (ii) the excess of this quantity is equal to the subentropy of the mixed states obtained by the partial trace of the bipartite pure states; and finally, (iii) these functionals can be considered as alternative measures of entanglement. Here, our intention is to establish an additional measure of entanglement (limited to the interval $[0,1]$) for Wehrl's entropy functionals described by bipartite Husimi functions which admit $\mathfrak{su}(1,1) \times \mathfrak{su}(1,1)$ coherent-state representations.